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Strong Uniform Approximation by Double Fourier Series

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We study the rate of strong uniform approximation to continuous functions $f(x, y)$, 2π-periodic in each variable, by the rectangular partial sums of their double Fourier series. As special cases, we deduce strong approximation rates to functions in the Lipschitz classes $\mathrm{Lip}(\alpha, \beta)$ and Zygmund classes $Z(\alpha, \beta)$, where $\alpha, \beta \in (0, 1]$. We also obtain the rates of strong uniform approximation to the conjugate functions $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$, and $\tilde{f}^{(1,1)}$ by the rectangular partial sums of the corresponding conjugate series. With two exceptions, all rates are shown to be the best possible.

1. INTRODUCTION

Let $f(x, y)$ be a complex-valued function, 2π-periodic in each variable and integrable over the two-dimensional torus $(-\pi, \pi] \times (-\pi, \pi]$. We remind the reader that the double Fourier series of f is defined by

$$S[f] = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}, \tag{1.1}$$

where

$$c_{jk} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v) e^{-i(ju+kv)}\, du\, dv. \tag{1.2}$$

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We consider the symmetric rectangular partial sums

$$s_{mn}(f, x, y) = \sum_{j=-m}^{m} \sum_{k=-n}^{n} c_{jk}\, e^{i(jx+ky)} \qquad (m, n = 0, 1, ...)$$

of series (1.1). It follows from (1.2) that

$$s_{mn}(f, x, y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v)\, D_n(u)\, D_n(v)\, du\, dv, \tag{1.3}$$

where $D_m(u)$ and $D_n(v)$ are the Dirichlet kernels in terms of u and v, respectively.

For the definition of the three conjugate series $\tilde{S}^{(1,0)}[f]$, $\tilde{S}^{(0,1)}[f]$, $\tilde{S}^{(1,1)}[f]$ as well as the corresponding conjugate functions $\tilde{f}^{(1,0)}(x, y)$, $\tilde{f}^{(0,1)}(x, y)$, $\tilde{f}^{(1,1)}(x, y)$, we refer to our previous paper [6].

2. Moduli of Continuity and Smoothness

From now on, let $f(x, y)$ be a continuous function, 2π-periodic in each variable, in symbols $f \in C_{2\pi \times 2\pi}$.

In the sequel, δ_1 and δ_2 denote nonnegative real numbers. The (total) modulus of continuity of f is defined by

$$\omega_1(f, \delta_1, \delta_2) = \sup_{|u| \leqslant \delta_1, |v| \leqslant \delta_2} \max_{(x, y)} |f(x+u, y+v) - f(x, y)|,$$

while

$$\omega_{1,x}(f, \delta_1) = \omega_1(f, \delta_1, 0) \quad \text{and} \quad \omega_{1,y}(f, \delta_2) = \omega_1(f, 0, \delta_2)$$

are called the partial moduli of continuity. For α, $\beta \in (0, 1]$, the Lipschitz class $\mathrm{Lip}(\alpha, \beta)$ is defined by

$$\mathrm{Lip}(\alpha, \beta) = \{f \in C_{2\pi \times 2\pi} : \omega_{1,x}(f, \delta_1) = \mathcal{O}\{\delta_1^\alpha\} \text{ and } \omega_{1,y}(f, \delta_2) = \mathcal{O}\{\delta_2^\beta\}\}.$$

The (total) modulus of symmetric smoothness of f is defined by

$$\omega_2(f, \delta_1, \delta_2) = \sup_{|u| \leqslant \delta_1, |v| \leqslant \delta_2} \max_{(x, y)} |\varphi_{x,y}(u, v)|,$$

where

$$\varphi_{x,y}(u, v) = \tfrac{1}{4}[f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u, y-v) - 4f(x, y)], \tag{2.1}$$

while

$$\omega_{2,x}(f, \delta_1) = \omega_2(f, \delta_1, 0) \qquad \text{and} \qquad \omega_{2,y}(f, \delta_2) = \omega_2(f, 0, \delta_2)$$

are called the partial moduli of smoothness. For $\alpha, \beta \in (0, 2]$, the Zygmund class $Z(\alpha, \beta)$ is defined by

$$Z(\alpha, \beta) = \{f \in C_{2\pi \times 2\pi} : \omega_{2,x}(f, \delta_1) = \mathcal{O}\{\delta_1^\alpha\} \text{ and } \omega_{2,y}(f, \delta_2) = \mathcal{O}\{\delta_2^\beta\}\}.$$

As is known, $\text{Lip}(\alpha, \beta) = Z(\alpha, \beta)$ if $\max\{\alpha, \beta\} < 1$ and $\text{Lip}(\alpha, \beta) \subset Z(\alpha, \beta)$ if $\max\{\alpha, \beta\} = 1$.

Remark 1. Let ω denote either ω_1 or ω_2. Then, obviously,

$$\begin{aligned} \max\{\omega_x(f, \delta_1), \omega_y(f, \delta_2)\} &\leqslant \omega(f, \delta_1, \delta_2) \\ &\leqslant \omega_x(f, \delta_1) + \omega_y(f, \delta_2). \end{aligned} \tag{2.2}$$

In [8], another modulus of smoothness of f is defined by

$$\omega^*(f; \delta_1, \delta_2) = \sup_{|u| \leqslant \delta_1, |v| \leqslant \delta_2} \max_{(x, y)} \tfrac{1}{2} |f(x+u, y+v) + f(x-u, y-v) - 2f(x, y)|.$$

The deficiency of this definition is that the second inequality in (2.2) is no longer true if ω is replaced by ω^*. In fact, putting $f(x, y) = xy$ we can see that

$$\omega^*(f, \delta_1, \delta_2) = \delta_1 \delta_2,$$

while

$$\omega_{2,x}(f, \delta_1) = \omega^*(f, \delta_1, 0) = 0 \qquad \text{and} \qquad \omega_{2,y}(f, \delta_2) = \omega^*(f, 0, \delta_2) = 0.$$

On the other hand, Definition (2.1) is motivated by the representation

$$s_{mn}(f, x, y) - f(x, y) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \varphi_{x,y}(u, v) D_m(u) D_n(v) \, du \, dv,$$

which follows from (1.3).

3. Main Results: Approximation by Fourier Series

Let $\gamma, \delta > -1$ be real numbers. We shall consider the Cesàro means

$$\sigma_{mn}^{\gamma\delta}(f, x, y) = \frac{1}{A_m^\gamma A_n^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} s_{jk}(f, x, y)$$

of series (1.1), where

$$A_m^\gamma = \binom{\gamma+m}{m} = \frac{(\gamma+m)(\gamma+m-1)\cdots(\gamma+1)}{m!}$$

for $m=1, 2, \ldots$ and $A_m^\gamma = 1$ for $m=0$.

The strong approximation operator $H_{mn}^{\gamma\delta}(f, p)$ is defined by

$$H_{mn}^{\gamma\delta}(f, p, x, y) = \left\{ \frac{1}{A_m^\gamma A_n^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} |s_{jk}(f, x, y) - f(x, y)|^p \right\}^{1/p},$$

where $p>0$. By Hölder's inequality, $H_{mn}^{\gamma\delta}(f, p, x, y)$ is nondecreasing in p, and for $p=1$ clearly

$$\mathscr{T}_{mn}^{\gamma\delta}(f) = |\sigma_{mn}^{\gamma\delta}(f, x, y) - f(x, y)| \leqslant H_{mn}^{\gamma\delta}(f, 1, x, y). \tag{3.1}$$

Denote by $E_{mn}(f)$ the best uniform approximation to f by two-dimensional trigonometric polynomials $t_{mn}(x, y)$ of degree $\leqslant m$ with respect to x and of degree $\leqslant n$ with respect to y,

$$E_{mn}(f) = \inf_{\{t_{mn}\}} \|t_{mn}(x, y) - f(x, y)\|,$$

where $\|\cdot\|$ is the usual maximum norm $\|\cdot\|_{C_{2\pi\times 2\pi}}$ henceforth.

The following theorem is an extension of a theorem by the second named author [7] (see also [5]) from one-dimensional to two-dimensional Fourier series.

THEOREM 1. *If $f \in C_{2\pi\times 2\pi}$ and $\gamma, \delta, p > 0$, then*

$$\|H_{mn}^{\gamma\delta}(f, p)\| = \mathcal{O}\left\{ \frac{1}{A_m^\gamma A_n^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} [E_{jk}(f)]^p \right\}^{1/p}. \tag{3.2}$$

The particular case $\gamma=\delta=1$ was announced by Gogoladze [1].

We refer to the extension of the classical Jackson theorem to continuous functions in two variables.

PROPOSITION 1. *If $f \in C_{2\pi\times 2\pi}$, then*

$$E_{mn}(f) = \mathcal{O}\left\{ \omega_{2,x}\left(f, \frac{1}{m+1}\right) + \omega_{2,y}\left(f, \frac{1}{n+1}\right) \right\}. \tag{3.3}$$

Theorem 1 and Proposition 1 yield the following.

COROLLARY 1. *If $f \in Z(\alpha, \beta)$, $\alpha, \beta \in (0, 1]$, and $\gamma, \delta, p > 0$, then*

$$\|H_{mn}^{\gamma\delta}(f, p)\| = \begin{cases} \mathcal{O}\left\{\dfrac{1}{(m+1)^{\alpha}} + \dfrac{1}{(n+1)^{\beta}}\right\} \\ \quad \text{if } \alpha p < 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+2)]^{1/p}}{(m+1)^{1/p}} + \dfrac{1}{(n+1)^{\beta}}\right\} \\ \quad \text{if } \alpha p = 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+1)]^{1/p}}{(m+1)^{1/p}} + \dfrac{[\log(n+1)]^{1/p}}{(n+1)^{1/p}}\right\} \\ \quad \text{if } \alpha p = \beta p = 1, \\ \mathcal{O}\left\{\dfrac{1}{(m+1)^{1/p}} + \dfrac{1}{(n+1)^{\beta}}\right\} \\ \quad \text{if } \alpha p > 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{1}{(m+1)^{1/p}} + \dfrac{[\log(n+2)]^{1/p}}{(n+1)^{1/p}}\right\} \\ \quad \text{if } \alpha p > 1 \text{ and } \beta p = 1, \\ \mathcal{O}\left\{\dfrac{1}{(m+1)^{1/p}} + \dfrac{1}{(n+1)^{1/p}}\right\} \\ \quad \text{if } \alpha p > 1 \text{ and } \beta p > 1. \end{cases} \tag{3.4}$$

The three remaining cases, $\alpha p < 1$ and $\beta p = 1$, $\alpha p < 1$ and $\beta p > 1$, and $\alpha p = 1$ and $\beta p > 1$, are the symmetric counterparts of (3.4)(ii), (iv), and (v), respectively.

The approximation rates in (3.4) are the best possible. To go into details, denote by $\{\lambda(n)\colon n = 0, 1, ...\}$ an arbitrary sequence of positive numbers converging to zero.

PROPOSITION 2. *There exist functions $f = f_{\alpha} \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha \leqslant 1$, such that for all γ, δ, $p > 0$, the estimates*

$$H_{mn}^{\gamma\delta}(f, p, 0, 0) = \begin{cases} o\left\{\dfrac{1}{(m+1)^{\alpha}}\right\} + \mathcal{O}\{\lambda(n)\} & \text{if } \alpha p < 1, \\ o\left\{\dfrac{[\log(m+2)]^{1/p}}{(m+1)^{1/p}}\right\} + \mathcal{O}\{\lambda(n)\} & \text{if } \alpha p = 1, \\ o\left\{\dfrac{1}{(m+1)^{1/p}}\right\} + \mathcal{O}\{\lambda(n)\} & \text{if } \alpha p > 1 \end{cases} \tag{3.5}$$

cannot hold.

These easily follow from the corresponding counterexamples constructed by Leindler [2, 3] in the case of one-dimensional Fourier series.

Remark 2. (i) By (3.3)–(3.5) we can see that for $f \in Z(\alpha, \beta)$, $\|H_{mn}^{\gamma\delta}(f, p)\|$ has the same order of magnitude as $E_{mn}(f)$ does if $\max\{\alpha p, \beta p\} < 1$, while the order of $\|H_{mn}^{\gamma\delta}(f, p)\|$ becomes worse than that of $E_{mn}(f)$ if $\max\{\alpha p, \beta p\} \geq 1$.

(ii) A trivial consequence of (3.1) and (3.2) is that if $f \in C_{2\pi \times 2\pi}$ and γ, $\delta > 0$, then

$$\mathscr{T}_{mn}^{\gamma\delta}(f) = \mathcal{O}\left\{\frac{1}{A_m^\gamma A_n^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1} E_{jk}(f)\right\}.$$

A comparison of Corollary 1 and [6, Theorem 3] shows that for $f \in Z(\alpha, \beta)$ the order of $\|H_{mn}^{\gamma\delta}(f, 1)\|$ is not worse than that of $\mathscr{T}_{mn}^{\gamma\delta}(f)$ including the cases where $\max\{\alpha, \beta\} = 1$.

However, this phenomenon is no longer true if we consider approximation to the conjugate functions. For instance, for $f \in \text{Lip}(1, \beta)$ the order of $\mathscr{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ is better than that of $\|H_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}, 1)\|$. (See Remark 3 in Section 4 below.)

(iii) Similarly to the one-dimensional case, generally speaking there is no difference between the classes $\text{Lip}(\alpha, \beta)$ and $Z(\alpha, \beta)$ as to the order of $\|H_{mn}^{\gamma\delta}(f, p)\|$.

4. Application: Approximation by Conjugate Series

The following auxiliary result proved in [6] plays a key role in this Section.

Lemma A. *If* $f \in Z(\alpha, \beta)$ *and* $0 < \alpha$, $\beta \leq 1$, *then*

$$\omega_{2,x}(\tilde{f}^{(1,0)}, \delta) = \mathcal{O}\{\delta^\alpha\},$$

$$\omega_{2,y}(\tilde{f}^{(1,0)}, \delta) = \mathcal{O}\left\{\delta^\beta \log \frac{1}{\delta}\right\},$$

$$\omega_{2,x}(\tilde{f}^{(1,1)}, \delta) = \mathcal{O}\left\{\delta^\alpha \log \frac{1}{\delta}\right\}.$$

Now combining Theorem 1 and Lemma A yields the following two corollaries.

COROLLARY 2. *If* $f \in Z(\alpha, \beta)$, $\alpha, \beta \in (0, 1]$, *and* $\gamma, \delta, p > 0$, *then*

$$\|H_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}, p)\| = \begin{cases} \mathcal{O}\left\{\dfrac{1}{(m+1)^{\alpha}} + \dfrac{\log(n+2)}{(n+1)^{\beta}}\right\} & \text{if } \alpha p < 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+2)]^{1/p}}{(m+1)^{1/p}} + \dfrac{\log(n+2)}{(n+1)^{\beta}}\right\} & \text{if } \alpha p = 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{1}{(m+1)^{\alpha}} + \dfrac{[\log(n+2)]^{(p+1)/p}}{(n+1)^{1/p}}\right\} & \text{if } \alpha p < 1 \text{ and } \beta p = 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+2)]^{1/p}}{(m+1)^{1/p}} + \dfrac{[\log(n+2)]^{(p+1)/p}}{(n+1)^{1/p}}\right\} & \text{if } \alpha p = \beta p = 1. \end{cases} \tag{4.1}$$

In the cases where $\max\{\alpha p, \beta p\} > 1$, we have estimates analogous to those in (3.4)(iv), (v), and (vi). The same remark pertains to Corollary 3 below. Furthermore, the corresponding estimates for $\|H_{mn}^{\gamma\delta}(\tilde{f}^{(0,1)}, p)\|$ are the symmetric counterparts of those in (4.1).

COROLLARY 3. *If* $f \in Z(\alpha, \beta)$, $\alpha, \beta \in (0, 1]$, *and* $\gamma, \delta, p > 0$, *then*

$$\|H_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}, p)\| = \begin{cases} \mathcal{O}\left\{\dfrac{\log(m+2)}{(m+1)^{\alpha}} + \dfrac{\log(n+2)}{(n+1)^{\beta}}\right\} & \text{if } \alpha p < 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+2)]^{(p+1)/p}}{(m+1)^{1/p}} + \dfrac{\log(n+2)}{(n+1)^{\beta}}\right\} & \text{if } \alpha p = 1 \text{ and } \beta p < 1, \\ \mathcal{O}\left\{\dfrac{[\log(m+2)]^{(p+1)/p}}{(m+1)^{1/p}} + \dfrac{[\log(n+2)]^{(p+1)/p}}{(n+1)^{1/p}}\right\} & \text{if } \alpha p = \beta p = 1. \end{cases} \tag{4.2}$$

Now we turn to the question of whether the approximation rates in Corollaries 2 and 3 are the best possible. Here $\{\lambda(n)\}$ again means an arbitrary sequence of positive numbers converging to zero.

PROPOSITION 3. *There exist functions $f = f_\alpha \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha \leqslant 1$, such that for γ, δ, $p > 0$, the estimates*

$$H_{mn}^{\gamma\delta}(\mathcal{T}^{(1,0)}, p, 0, 0) = \begin{cases} o\left\{\dfrac{1}{(m+1)^\alpha}\right\} + \mathcal{O}\{\lambda(n)\} & \text{if } \alpha p < 1, \\ o\left\{\dfrac{[\log(m+2)]^{1/p}}{(m+1)^{1/p}}\right\} + \mathcal{O}\{\lambda(n)\} & \text{if } \alpha p = 1 \end{cases} \tag{4.3}$$

cannot hold.

Furthermore, there exist functions $f = f_\beta \in \mathrm{Lip}(1, \beta)$, $0 < \beta \leqslant 1$, such that for γ, $\delta > 0$ and $p \geqslant 1$, the estimates

$$H_{mn}^{\gamma\delta}(\mathcal{T}^{(1,0)}, p, 0, 0) = \begin{cases} \mathcal{O}\{\lambda(m)\} + o\left\{\dfrac{\log(n+2)}{(n+1)^\beta}\right\} & \text{if } \beta p < 1, \\ \mathcal{O}\{\lambda(m)\} + o\left\{\dfrac{[\log(n+2)]^2}{n+1}\right\} & \text{if } p = \beta = 1 \end{cases} \tag{4.4}$$

cannot hold.

In fact, (4.3) is identical with (3.5)(i) and (ii) applied for $\mathcal{T}^{(1,0)}$ in place of f, while (4.4) follows from [6, Theorem 6] via (3.1) and Hölder's inequality. In the cases where $\alpha p > 1$ or $\beta p > 1$, we have counterexamples analogous to those in (3.5)(iii). The same remark applies to Proposition 4 below.

The only rates in (4.1) we are unable to prove to be the best possible are the second halves of (iii) and (iv) for $0 < \beta < 1$ and $\beta p = 1$.

Conjecture 1. There exist functions $f = f_\beta \in \mathrm{Lip}(1, \beta)$, $0 < \beta < 1$, such that for γ, $\delta > 0$ and $p = 1/\beta$, the estimate

$$H_{mn}^{\gamma\delta}(\mathcal{T}^{(1,0)}, p, 0, 0) = \mathcal{O}\{\lambda(m)\} + o\left\{\frac{[\log(n+2)]^{(p+1)/p}}{(n+1)^{1/p}}\right\} \tag{4.5}$$

cannot hold.

Clearly, (4.5) for $\beta = 1$ coincides with (4.4)(ii).

PROPOSITION 4. *There exist functions $f = f_\alpha \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha < 1$, such that for γ, $\delta > 0$ and $1 \leqslant p < 1/\alpha$, the estimate*

$$H_{mn}^{\gamma\delta}(\mathcal{T}^{(1,1)}, p, 0, 0) = o\left\{\frac{\log(m+2)}{(m+1)^\alpha}\right\} + \mathcal{O}\{\lambda(n)\} \tag{4.6}$$

cannot hold.

Indeed, (4.6) follows from [6, Theorem 7] via (3.1) and Hölder's inequality.

According to Proposition 4, the rates in (4.2)(i) and in the second half of (4.2)(ii) are the best possible. In the cases where $\alpha p = 1$ or $\beta p = 1$ we formulate the following.

Conjecture 2. There exist functions $f = f_\alpha \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha \leqslant 1$, such that for γ, $\delta > 0$ and $p = 1/\alpha$, the estimate

$$H_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}, p, 0, 0) = o\left\{\frac{[\log(m+2)]^{(p+1)/p}}{(m+1)^{1/p}}\right\} + \mathcal{O}\{\lambda(n)\}$$

cannot hold.

Remark 3. A comparison with the results of [6] shows that for $f \in \mathrm{Lip}(\alpha, \beta)$, $\|H_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)}, 1)\|$ has the same order of magnitude as $\mathscr{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,0)})$ only in the cases of (4.1)(i) and (iii), that is when $\alpha < 1$. Furthermore, for $f \in \mathrm{Lip}(\alpha, \beta)$, $\|H_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)}, 1)\|$ has the same order as $\mathscr{T}_{mn}^{\gamma\delta}(\tilde{f}^{(1,1)})$ only in the cases of (4.2)(i) and of the second half of (4.2)(ii), that is when $\max\{\alpha, \beta\} < 1$. Otherwise (i.e., where $\alpha = 1$ or $\max\{\alpha, \beta\} = 1$, respectively), the order of $\|H_{mn}^{\gamma\delta}(\cdot, 1)\|$ is worse than that of $\mathscr{T}_{mn}^{\gamma\delta}(\cdot)$.

5. Another Strong Approximation Operator

Analogous results can be proved for the operator

$$K_{mn}^{\gamma\delta}(f, p, x, y) = \left\{\frac{1}{(m+1)^\gamma (n+1)^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} (j+1)^{\gamma-1} \times (k+1)^{\delta-1} |s_{jk}(f, x, y) - f(x, y)|^p\right\}^{1/p}.$$

THEOREM 2. *If $f \in C_{2\pi \times 2\pi}$ and γ, δ, $p > 0$, then*

$$\|K_{mn}^{\gamma\delta}(f, p)\| = \mathcal{O}\left\{\frac{1}{(m+1)^\gamma (n+1)^\delta} \sum_{j=0}^{m} \sum_{k=0}^{n} (j+1)^{\gamma-1} \times (k+1)^{\delta-1} [E_{jk}(f)]^p\right\}^{1/p}. \tag{5.1}$$

This is an extension of a result of Leindler (see [2, 4]) from one-dimensional to two-dimensional Fourier series.

On the basis of Theorem 2 and Proposition 1 we can deduce consequences similar to Corollaries 2–4 and the rates obtained are also the best possible in the cases indicated in Propositions 2–4.

6. Auxiliary Results

We begin with a lemma of Leindler [2] for one-dimensional Fourier series. Given an integrable function g in one variable, we denote by $s_j(g, x)$ the symmetric partial sums of the Fourier series of g, where $j = 0, 1, \ldots$.

Lemma B. *If $g \in C_{2\pi}$ and $p > 0$, then*

$$\left\| \left\{ \frac{1}{m+1} \sum_{j=0}^{m} |s_j(g, x)|^p \right\}^{1/p} \right\| = \mathcal{O}\{\|g\|\},$$

where "$\mathcal{O}$" depends only on p.

We extend this to two-dimensional Fourier series as follows.

Lemma 1. *If $f \in C_{2\pi \times 2\pi}$ and $p > 0$, then*

$$\begin{aligned} \Sigma_p &= \Sigma_p(f, m, n) \\ &= \left\| \left\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} |s_{jk}(f, x, y)|^p \right\}^{1/p} \right\| \\ &= \mathcal{O}\{\|f\|\}, \end{aligned} \tag{6.1}$$

where "$\mathcal{O}$" depends only on p.

Proof. Since Σ_p is nondecreasing in p (for fixed f, m, n), we may assume that $p \geqslant 2$. We put

$$I_m = \{u\colon |u| \leqslant 1/(m+1)\}, \qquad J_n = \{v\colon |v| \leqslant 1/(n+1)\},$$

and for their complements to $[-\pi, \pi]$

$$CI_m = \{u\colon 1/(m+1) < |u| \leqslant \pi\}, \qquad CJ_n = \{v\colon 1/(n+1) < |v| \leqslant \pi\}.$$

We split the double integral in Representation (1.3) as follows:

$$\begin{aligned} s_{jk}(f, x, y) = \frac{1}{\pi^2} \left\{ \int_{I_m} \int_{J_n} + \int_{I_m} \int_{CJ_n} + \int_{CI_m} \int_{J_n} + \int_{CI_m} \int_{CJ_n} \right\} \\ \times f(x+u, y+v)\, D_j(u)\, D_k(v)\, du\, dv. \end{aligned} \tag{6.2}$$

Denote by $\Sigma_p^{(1)}$, $\Sigma_p^{(2)}$, $\Sigma_p^{(3)}$, and $\Sigma_p^{(4)}$ the corresponding quantities defined analogously to (6.1) by substituting the subintegrals in (6.2) for $s_{jk}(f, x, y)$. For example,

$$\begin{aligned} \Sigma_p^{(1)} = \Bigg\| \Bigg\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} \Bigg| \frac{1}{\pi^2} \int_{I_m} \int_{J_n} f(x+u, y+v) \\ \times D_j(u)\, D_k(v)\, du\, dv \Bigg|^p \Bigg\}^{1/p} \Bigg\|. \end{aligned}$$

We are going to show that the order of magnitude of each $\Sigma_p^{(l)}$ is $\mathcal{O}\{\|f\|\}$. First, using the trivial inequalities

$$|D_j(u)| \leqslant j+1 \qquad \text{and} \qquad |D_k(v)| \leqslant k+1, \tag{6.3}$$

we get

$$\Sigma_p^{(1)} \leqslant \frac{4\,\|f\|}{\pi^2} \left\{ \frac{1}{(m+1)^{p+1}(n+1)^{p+1}} \sum_{j=0}^{m} \sum_{k=0}^{n} (j+1)^p (k+1)^p \right\}^{1/p}$$
$$= \mathcal{O}\{\|f\|\}. \tag{6.4}$$

Second, by Fubini's theorem

$$\Sigma_p^{(2)} \leqslant \left\| \left\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} \left(\frac{1}{\pi^2} \int_{I_m} |D_j(u)|\, du \right.\right.\right.$$
$$\left.\left.\left. \times \left| \int_{CJ_n} f(x+u, y+v)\, D_k(v)\, dv \right| \right)^p \right\}^{1/p} \right\|.$$

Then we apply Jensen's inequality (see, e.g., [9, Vol. I, p. 24]) to the inner integral to obtain

$$\Sigma_p^{(2)} \leqslant \left\| \left\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} \left(\frac{1}{\pi} \int_{I_m} |D_j(u)|\, du \right)^{p-1} \right.\right.$$
$$\times \frac{1}{\pi} \int_{I_m} |D_j(u)|\, du$$
$$\left.\left. \times \left(\frac{1}{\pi} \left| \int_{CJ_n} f(x+u, y+v)\, D_k(v)\, dv \right| \right)^p \right\}^{1/p} \right\|.$$

Next by (6.3)(i) and Lemma B, we can conclude that

$$\Sigma_p^{(2)} \leqslant \left\| \left\{ \frac{1}{m+1} \sum_{j=0}^{m} \left(\frac{1}{\pi} \int_{I_m} (j+1)\, du \right)^{p-1} \frac{1}{\pi} \int_{I_m} (j+1)\, du \right.\right.$$
$$\left.\left. \times \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{1}{\pi} \left| \int_{CJ_n} f(x+u, y+v)\, D_k(v)\, dv \right| \right)^p \right\}^{1/p} \right\|$$
$$\leqslant \mathcal{O}\{\|f\|\} \left\{ \frac{1}{(m+1)^{p+1}} \sum_{j=0}^{m} (j+1)^p \right\}^{1/p}$$
$$= \mathcal{O}\{\|f\|\}. \tag{6.5}$$

Third, we can similarly derive that

$$\Sigma_p^{(3)} = \mathcal{O}\{\|f\|\}. \tag{6.6}$$

Fourth, using the representation

$$D_j(u)=\frac{\sin ju}{2\tan(1/2)u}+\frac{1}{2}\cos ju$$

and an analogous one for $D_k(v)$, we can estimate as follows:

$$\begin{aligned}\Sigma_p^{(4)} &\leqslant \left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{\pi^2}\left|\int_{CI_m}\int_{CJ_n} f(x+u,\, y+v)\right.\right.\right.\right.\\ &\qquad \left.\left.\left.\left.\times \tfrac{1}{4}\cos ju\cos kv\, du\, dv\right|\right)^p\right\}^{1/p}\right\|\\ &\quad+\left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{\pi^2}\left|\int_{CI_m}\int_{CJ_n} f(x+u,\, y+v)\right.\right.\right.\right.\\ &\qquad \left.\left.\left.\left.\times \frac{1}{2}\cos ju\,\frac{\sin kv}{2\tan(1/2)v}\, du\, dv\right|\right)^p\right\}^{1/p}\right\|\\ &\quad+\left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{\pi^2}\left|\int_{CI_m}\int_{CJ_n} f(x+u,\, y+v)\right.\right.\right.\right.\\ &\qquad \left.\left.\left.\left.\times \frac{\sin ju}{2\tan(1/2)u}\,\frac{1}{2}\cos kv\, du\, dv\right|\right)^p\right\}^{1/p}\right\|\\ &\quad+\left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{\pi^2}\left|\int_{CI_m}\int_{CJ_n} f(x+u,\, y+v)\right.\right.\right.\right.\\ &\qquad \left.\left.\left.\left.\times \frac{\sin ju}{2\tan(1/2)u}\,\frac{\sin kv}{2\tan(1/2)v}\, du\, dv\right|\right)^p\right\}^{1/p}\right\|\\ &=\Sigma_p^{(41)}+\Sigma_p^{(42)}+\Sigma_p^{(43)}+\Sigma_p^{(44)}, \text{ say.}\end{aligned} \tag{6.7}$$

(i) Clearly,

$$\Sigma_p^{(41)}\leqslant \|f\|. \tag{6.8}$$

(ii) By Jensen's inequality,

$$\begin{aligned}\Sigma_p^{(42)} &\leqslant \left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\int_{CI_m}\frac{du}{2\pi}\right.\right.\right.\\ &\qquad \left.\left.\left.\times\left|\frac{1}{\pi}\int_{CJ_n} f(x+u,\, y+v)\,\frac{\sin kv}{2\tan(1/2)v}\, dv\right|\right)^p\right\}^{1/p}\right\|\\ &\leqslant \left\|\left\{\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}\left(\int_{CI_m}\frac{du}{2\pi}\right)^{p-1}\right.\right.\\ &\qquad \left.\left.\times\int_{CI_m}\frac{du}{2\pi}\left(\frac{1}{\pi}\left|\int_{CJ_n} f(x+u,\, y+v)\,\frac{\sin kv}{2\tan(1/2)v}\, dv\right|\right)^p\right\}^{1/p}\right\|.\end{aligned}$$

Since $p \geqslant 2$ we can apply the Hausdorff–Young inequality (see, e.g. [9, Vol. II, p. 101]) to obtain

$$\Sigma_p^{(42)} \leqslant \left\| \left\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \int_{CI_m} \frac{du}{2\pi} \times \left\{ \int_{CJ_n} \frac{|f(x+u, y+v)|^q}{|2\tan(1/2)v|^q} dv \right\}^{p/q} \right\}^{1/p} \right\|,$$

where the conjugate exponent q is defined by $1/p + 1/q = 1$. Hence

$$\Sigma_p^{(42)} \leqslant \left\{ \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \mathcal{O}\{\|f\|^p\} \left(2 \int_{1/(n+1)}^{\pi} \frac{dv}{v^q} \right)^{p/q} \right\}^{1/p}$$
$$\leqslant \left\{ \frac{\mathcal{O}\{\|f\|^p\}}{n+1} (n+1)^{(q-1)p/q} \right\}^{1/p} = \mathcal{O}\{\|f\|\}. \tag{6.9}$$

(iii) Analogously,

$$\Sigma_p^{(43)} = \mathcal{O}\{\|f\|\}. \tag{6.10}$$

(iv) Finally, applying the Hausdorff–Young inequality extended to two-dimensional Fourier series, we find that

$$\Sigma_p^{(44)} \leqslant \left\| \left\{ \frac{1}{(m+1)(n+1)} \times \mathcal{O} \left\{ \int_{CI_m} \int_{CJ_n} \frac{|f(x+u, y+v)|^q}{|4\tan(1/2)u\tan(1/2)v|^q} du\, dv \right\}^{p/q} \right\}^{1/p} \right\|$$
$$\leqslant \left\{ \frac{\mathcal{O}\{\|f\|^p\}}{(m+1)(n+1)} \left(4 \int_{1/(m+1)}^{\pi} \int_{1/(n+1)}^{\pi} \frac{du\, dv}{u^q v^q} \right)^{p/q} \right\}^{1/p}$$
$$= \mathcal{O}\{\|f\|\}, \tag{6.11}$$

in a similar manner as in the case of (ii).

Putting (6.7)–(6.11) together yields

$$\Sigma_p^{(4)} = \mathcal{O}\{\|f\|\}. \tag{6.12}$$

Combining (6.2), (6.4)–(6.6), and (6.12) furnishes (6.1), which was to be proved.

The following consequence of Lemma 1 plays a key role in the proofs of Theorems 1 and 2.

LEMMA 2. *If $f \in C_{2\pi \times 2\pi}$ and $p > 0$, then for $m, n = 1, 2, \ldots$*

$$\left\| \left\{ \frac{1}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |s_{jk}(f, x, y) - f(x, y)|^p \right\}^{1/p} \right\| = \mathcal{O}\{E_{mn}(f)\}, \tag{6.13}$$

$$\left\| \left\{ \frac{1}{m} \sum_{j=m}^{2m-1} |s_{jo}(f, x, y) - f(x, y)|^p \right\}^{1/p} \right\| = \mathcal{O}\{E_{mo}(f)\}, \tag{6.14}$$

$$\left\| \left\{ \frac{1}{n} \sum_{k=n}^{2n-1} |s_{ok}(f, x, y) - f(x, y)|^p \right\}^{1/p} \right\| = \mathcal{O}\{E_{on}(f)\}, \tag{6.15}$$

and

$$\|s_{oo}(f, x, y) - f(x, y)\| = \mathcal{O}\{E_{oo}(f)\}, \tag{6.16}$$

where the "$\mathcal{O}$"s depend only on p.

Proof. Denote by $t^*_{mn}(f, x, y)$ the trigonometric polynomial of degree $\leqslant m$ with respect to x and of degree $\leqslant n$ with respect to y such that

$$\|f(x, y) - t^*_{mn}(f, x, y)\| = E_{mn}(f)$$

and denote by $\Sigma^*_p(f, m, n)$ the left-hand side of (6.13).

If $p \geqslant 1$, then by Hölder's inequality and (6.1),

$$\begin{aligned} \Sigma^*_p(f, m, n) &\leqslant 2^{2/p} \Sigma_p(f - t^*_{mn}(f), 2m-1, 2n-1) \\ &\quad + \left\| \left\{ \frac{1}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |t^*_{jk}(f, x, y) - f(x, y)|^p \right\}^{1/p} \right\| \\ &\leqslant 2^{2/p} \mathcal{O}\{\|f - t^*_{mn}(f)\|\} + \left\{ \frac{1}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} [E_{jk}(f)]^p \right\}^{1/p} \\ &= \mathcal{O}\{E_{mn}(f)\}, \end{aligned}$$

proving (6.13).

If $0 < p \leqslant 1$, then

$$\begin{aligned} [\Sigma^*_p(f, m, n)]^p &\leqslant 4[\Sigma_p(f - t^*_{mn}(f), 2m-1, 2n-1)]^p \\ &\quad + \left\| \frac{1}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |t^*_{jk}(f, x, y) - f(x, y)|^p \right\|, \end{aligned}$$

which implies (6.13) in the same way as above.

The proofs of (6.14)–(6.16) follow a similar pattern, so we omit them.

7. Proofs of Theorems 1 and 2

The two proofs are much alike, but the proof of Theorem 2 is technically simpler.

Proof of Theorem 2. We may assume $m, n \geqslant 1$. Otherwise, Theorem 2 reduces to the corresponding one-dimensional result. In the sequel, let q and r be positive integers such that

$$2^q \leqslant m+1 < 2^{q+1} \qquad \text{and} \qquad 2^r \leqslant n+1 < 2^{r+1}.$$

We distinguish four cases according to $\gamma \geqslant 1$ or $0<\gamma<1$ and $\delta \geqslant 1$ or $0<\delta<1$.

Case 1. $\gamma, \delta \geqslant 1$. Then both $(j+1)^{\gamma-1}$ and $(k+1)^{\delta-1}$ are nondecreasing. An elementary estimation and Lemma 2 give that

$$\begin{aligned}
&(m+1)^{\gamma}(n+1)^{\delta}\,[K^{\gamma\delta}_{mn}(f,p)]^p \\
&\leqslant |s_{oo}(f)-f|^p + \sum_{l=1}^{q+1} 2^{(\gamma-1)l} \sum_{j=2^{l-1}}^{2^l-1} |s_{jo}(f)-f|^p \\
&\quad + \sum_{\bar{l}=1}^{r+1} 2^{(\delta-1)\bar{l}} \sum_{k=2^{\bar{l}-1}}^{2^{\bar{l}}-1} |s_{ok}(f)-f|^p \\
&\quad + \sum_{l=1}^{q+1} \sum_{\bar{l}=1}^{r+1} 2^{(\gamma-1)l} 2^{(\delta-1)\bar{l}} \sum_{j=2^{l-1}}^{2^l-1} \sum_{k=2^{\bar{l}-1}}^{2^{\bar{l}}-1} |s_{jk}(f)-f|^p \\
&= C\left\{E^p_{oo} + 2\sum_{l=1}^{q+1} 2^{\gamma l} E^p_{2^{l-1},o} + 2\sum_{\bar{l}=1}^{r+1} 2^{\delta\bar{l}} E^p_{o,2^{\bar{l}-1}} + 4\sum_{l=1}^{q+1}\sum_{\bar{l}=1}^{r+1} 2^{\gamma l} 2^{\delta\bar{l}} E^p_{2^{l-1},2^{\bar{l}-1}}\right\},
\end{aligned} \tag{7.1}$$

where $f=f(x,y)$, $s_{jk}(f)=s_{jk}(f,x,y)$, $E_{jk}=E_{jk}(f)$, etc.

On the other hand, using the nonincreasing property of E_{jk} in j and k, an easy calculation yields

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} (j+1)^{\gamma-1}(k+1)^{\delta-1} E^p_{jk} \\
&\qquad \geqslant E^p_{oo} + 2^{-\gamma}\sum_{l=1}^{q} 2^{\gamma l} E^p_{2^l,o} + 2^{-\delta}\sum_{\bar{l}=1}^{r} 2^{\delta\bar{l}} E^p_{o,2^{\bar{l}}} \\
&\qquad\quad + 2^{-\gamma}2^{-\delta}\sum_{l=1}^{q}\sum_{\bar{l}=1}^{r} 2^{\gamma l}2^{\delta\bar{l}} E^p_{2^l,2^{\bar{l}}}.
\end{aligned} \tag{7.2}$$

Comparing the right-hand sides of (7.1) and (7.2) results in (5.1), which was to be proved.

Case 2. $0<\gamma,\ \delta<1$. Then both $(j+1)^{\gamma-1}$ and $(k+1)^{\delta-1}$ are non-increasing. Similarly to (7.1), we can conclude that

$$(m+1)^{\gamma}(n+1)^{\delta}[K_{mn}^{\gamma\delta}(f,p)]^{p}$$
$$=\mathcal{O}\left\{E_{oo}^{p}+2^{-\gamma}\sum_{l=1}^{q+1}2^{\gamma l}E_{2^{l-1},o}^{p}+2^{-\delta}\sum_{l=1}^{r+1}2^{\delta l}E_{o,2^{l-1}}^{p}\right.$$
$$\left.+2^{-\gamma}2^{-\delta}\sum_{l=1}^{q+1}\sum_{\bar{l}=1}^{r+1}2^{\gamma l}2^{\delta\bar{l}}E_{2^{l-1},2^{\bar{l}-1}}^{p}\right\}, \tag{7.3}$$

and similarly to (7.2),

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(j+1)^{\gamma-1}(k+1)^{\delta-1}E_{jk}^{p}\geqslant E_{oo}^{p}+\frac{1}{2}\sum_{l=1}^{q}2^{\gamma l}E_{2^{l},o}^{p}+\frac{1}{2}\sum_{l=1}^{r}2^{\delta l}E_{o,2^{l}}^{p}$$
$$+\frac{1}{4}\sum_{l=1}^{q}\sum_{\bar{l}=1}^{r}2^{\gamma l}2^{\delta\bar{l}}E_{2^{l},2^{\bar{l}}}^{p}. \tag{7.4}$$

Now, it suffices to employ the monotonicity property of $E_{jk}(f)$ in order to derive (5.1) on the basis of (7.3) and (7.4).

Case 3. $\gamma\geqslant 1$ and $0<\delta<1$.

Case 4. $0<\gamma<1$ and $\delta\geqslant 1$.

In the last two cases we can combine the estimation techniques applied in Cases (i) and (ii). We do not enter into details.

Proof of Theorem 1. This goes along essentially the same lines as the proof of Theorem 2. We have to keep in mind that for $\gamma>-1$ there exist two positive constants K_1 and K_2 depending only on γ such that

$$K_1(m+1)^{\gamma}\leqslant A_m^{\gamma}\leqslant K_2(m+1)^{\gamma} \qquad (m=0,1,\ldots)$$

(see, e.g., [9, Vol. I, p. 77]).

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